Robust consensus of fractional-order multi-agent systems with input saturation and external disturbances

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\section{Introduction}

Recently, distributed cooperative control of multi-agent systems has drawn increasing attention because of its broad application in numerous fields, such as UAVs, multi-robot cooperation and smart grids [1–3]. Multi-agent systems with distributed cooperative control can achieve multiple collective behavior, according to the control objectives, which includes consensus [4–11], formation control [12,13], synchronization [14–16], and flocking [17,18]. As consensus is the fundamental issue in cooperative control, it has been one of the most hot topic of cooperative control. In consensus control, each agent updates its state by interacting with its local neighbor agents to make all the agents arrive at the common state. The existing study on consensus primarily focuses on the integer-order systems, the dynamics of which are first-order, second-order even high-order differential equation.

At present, it has been shown that lots of natural phenomena cannot be effectively interpreted by using the integer-order kinetics, such as food searching of germs, chemotaxis behavior and the group movement of bacteria in lubrications secreted by themselves [19,20]. Besides, many practical systems cannot be described by integer-order systems when the agent works on some complicated environments [21–24], for example, aerial vehicles operating under the environment affected by the weather, automobiles running on the road-surface containing viscoelastic materials, and UAVs flying under a complicated environment with lots of particles. In virtue of their hereditary and memory property, fractional-order systems can describe the cases like the above more accurately. Therefore, more and more researches attach great importance to the distributed cooperative control of fractional-order multi-agent systems. Cao et al. [25] firstly investigated the distributed cooperation of fractional-order systems, where the cooperation algorithms were achieved by putting forward several sufficiency conditions. Gong [4] considered the nonlinear fractional-order multi-agent systems, leaderless and leader-follower consensus of which was achieved by using adaptive algorithm. Chen et.al [6] investigated the uncertain fractional-order multi-agent systems, containment consensus of which was achieved by using the analysis of stability and the theory of matrixes. Yang et.al [8] focused on the time-delayed fractional-order multi-agent systems, containment consensus of which was achieved by using frequency domain theorem and Laplace transform. Wang and Yang [26] considered the nonlinear fractional-order multi-agent systems, leader-follower consensus of which can be achieved and sufficiency conditions were presented for achieving consensus of such a system.

In practical applications, the capability of multi-agent systems is always affected by the disturbances [3,21,27–29]. Therefore, it's inevitable for many industries to consider the influence of disturbances. Take multi-robot systems as an example. In multi-robot...
system [21], the external disturbances such as friction of robot arm, electromagnetic interference, can make the robot unable to work even result in instability of the multi-robot system. The influence of disturbances is also included in satellite formation [29] and power systems [3]. Therefore, it is important to consider disturbance in studying multi-agent systems. In consideration of the great application prospect, some researchers devoted themselves to the fractional-order multi-agent systems with disturbances as well as got some achievement. Liu et al. [31] considered uncertain fractional-order nonlinear systems with external disturbances where the fractional adaptive fuzzy controller was constructed for ensuring specified performance. Wang et al. [30] investigated the uncertain fractional-order chaotic systems with exogenous disturbance and put forward a new approach for achieving robust MPS. Ren and Yu [7] studied the fractional-order multi-agent systems subject to exogenous disturbance, where linear as well as non-linear system reached robust consensus by utilizing inequality methods and Mittag-Leffler function. Yang et al. [32] achieved distributed cooperation control for multi-agent systems with external disturbance through developing a state observer based on the disturbance observers.

In the aforementioned literature, they haven’t considered the case when the agent is subject to input saturation. In general, control input subject to the bounded saturation is very prevalent in practical applications [33–35]. For example, in spacecraft [33], control input saturation can give rise to the windup phenomenon where the output is inconsistent with the input so that it results in performance deterioration even instability of the system. Therefore, according to the practical demand, it is urgent for us to study the consensus problem for fractional-order multi-agent systems with input saturation. However, to the best of our knowledge, no one has focused on the fractional-order multi-agent systems with input saturation.

Motivated by the above discussion, the robust consensus problem of fractional-order multi-agent systems with input saturation and external disturbances is studied in this paper. It is the first time to investigate the consensus problem of the fractional-order multi-agent systems with input saturation. Note that the well-known Leibnitz rule is not applicable for fractional-order derivatives, inspired by [7], the Mittag-Leffler function and fractional-order Lyapunov direct method are introduced to overcome the difficulty. Low gain feedback technique is utilized to achieve the robust consensus of fractional-order multi-agent systems with input saturation and external disturbances and the leader-following consensus of fractional-order multi-agent systems with input saturation, respectively.

The remainder of this paper is organized as follows. The algebraic graph theory, Caputo fractional operator and Mittag-Leffler function are introduced to be a foundation in Section 2. The problem statement and the main results on robust consensus for fractional-order multi-agent systems with input saturation and external disturbances are given in Sections 3 and 4. Then, several simulations are given in Section 5. Lastly, some conclusion is obtained in Section 6.

Notation: $\mathbb{Z}^+$ refers to a set of positive integers. $\mathbb{R}^{n \times b}$ is a vector space of matrices containing all $a \times b$ matrices. $\mathbb{R}^e$ refers to c-dimensional Euclidean space. $\| \cdot \|$ and $\|\cdot\|_\infty$ express Euclidean norm and $\infty$-norm respectively. $\text{diag}(\alpha_1, \ldots, \alpha_N)$ defines the diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_N$. Symmetric positive definite matrix $A$ is a matrix where $A \succ 0$ and $A = A^T$ hold. $\lambda_{\text{min}}(A)$ (\lambda_{\text{max}}(A)) represents the minimum (maximum) eigenvalue of matrix $A$. $C \otimes D$ refers to the Kronecker product of matrices $C$ and $D$. $\text{sgn}(x)$ expresses the sign function of $x$.

2. Preliminaries

In this section, the concept of algebraic graph theory and several basic definitions of the Caputo operator are presented since they are the mathematical foundations of this paper. Moreover, some conclusions used for the proof of following theorem are obtained.

2.1. Algebraic graph theory

A directed graph $G = (V, E)$ is used for describing the communication graph in the multi-agent system containing $N$ agents. In this graph, $V = \{1, 2, \ldots, N\}$ is the set of $N$ nodes where agent $i$ is expressed by node $i$, and the edge set $E = \{(i, j) \in V \times V\}$ can be used to represent the neighboring relation among agents. The edges of a directed graph are directed, i.e., the edge $(i, j)$ can be seen as an ordered pair which starts at agent $i$ and ends at agent $j$. Moreover, agent $j$ is known as a child of agent $i$ and agent $i$ is termed as a parent of agent $j$. When $(i, j) \in E$, then node $i$ contains a self-loop. The set of neighbor agents of agent $i$ are described as $N_i = \{j: (i, j) \in E\}$. For each edge of graph, there exits a corresponding non-negative weight of edge such that the graph is weighted graph. Denote $A = (a_{ij})_{N \times N}$ as the adjacency matrix and when $(j, j) \in E$, $a_{jj} > 0$ or else $a_{jj} = 0$. Besides, $a_{ij} = 0$ when self-loop is not considered. The Laplacian matrix of the directed graph $G$ is defined as $L = D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_N)$ is the degree matrix of $G$ with $d_i = \sum_{j=1}^N a_{ij}$. Moreover, $L = \{l_{ij}\}_{N \times N}$ satisfies $l_{ij} = -a_{ij}$ when $i \neq j$ otherwise $l_{ii} = \sum_{j=1}^N a_{ij}$. A directed path from node $i$ to node $k$ is a series of ordered edges in the form of $(i_1, i_2, i_3, \ldots, i_{l-1}, i_l)$. When $i_j \in V$, $(i_j, i_{j+1}) \in E, j = 1, \ldots, k - 1$. A spanning tree is contained in a graph if there is at least one node $i$ so that for any other node $j$, a path from node $i$ to $j$ exists. Denote $\bar{G}$ on the set of nodes $V = \{0, 1, 2, \ldots, N\}$ as a graph which contains $N$ agents as well as a leader $0$. If agent $i$ can get the information from the leader $0$, then define $h_i = 1$, otherwise $h_i = 0$. Define $H = \text{diag}(h_1, h_2, \ldots, h_N).

Assumption 1 ([36]). Suppose that the directed graph $\dot{G}$ has a directed spanning tree with the leader as the root vertex. Then, $L + H > 0$ and each eigenvalue of $L + H$ has positive real part.

2.2. Caputo fractional derivative and Mittag–Leffler function

Definition 1 ([37]). For an integrable function $y$, the definition of the Caputo fractional derivative with order $\alpha$ is

$$\text{i}_\alpha D^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^t \frac{y(\tau)}{(t - \tau)^{n-\alpha}} d\tau,$$

where $\tau \geq t_0$ and for positive integer $n, \alpha \in (n - 1, n)$. Moreover, for $\alpha \in (0, 1)$

$$\text{i}_\alpha D^\alpha y(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^t \frac{y(\tau)}{(t - \tau)^{\alpha}} d\tau.$$

Definition 2 ([37]). The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{\text{i}_\alpha D^\alpha y(t); s\} = s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(t_0),$$

where $s$ is defined in Laplace domain.

Property 1. For two continuous function $f, g$, there exist

$$\text{i}_\alpha D^\alpha (a f + b g) = a \text{i}_\alpha D^\alpha f + b \text{i}_\alpha D^\alpha g.$$
and $i_0D_0^\alpha c = 0$ with constant $a$, $b$, and $c$.

**Definition 3** ([38]). The Mittag–Leﬄer function with two positive parameter $\mu$ and $v$ can be described as follows:

$$E_{\mu,v}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu + v)},$$

where $z$ is a complex number and for $b=1$, $E_{\mu,1}(z) = E_{\mu}(z)$.

**Definition 4** ([38]). The Laplace transform of Mittag–Leﬄer function with two positive parameter $m$ and $n$ is denoted by $\mathcal{L}\{z^{m-n}E_{m,n}(z^{m-n})\} = \frac{\sqrt{\sigma}}{\sqrt{2}m + \sqrt{\lambda}}, \quad \lambda \in \mathbb{R}, \quad \Re(s) \geq \frac{1}{2}$, where $s$ is the variables in the Laplace domain and $t$ represents time. $\mathcal{L}\{\cdot\}$ express its Laplace transform and real part, respectively.

The following Lemmas are important for the subsequent proof:

**Lemma 2** ([39]). For $0 < \alpha < 2$, $\beta \in \mathbb{R}$, $\alpha \pi/2 < \theta < \min\{\pi, \alpha \pi\}$, $d > 0$, the estimate of Mittag–Leﬄer is

$$|E_{d/\beta}(z)| \leq \frac{d}{1 + |z|}, \quad \arg(z) \in [\theta, \pi], \quad |z| \in [0, \infty),$$

where $\arg(\cdot)$ represents the argument of complex number.

**Lemma 3** ([40]). For a continuously differentiable vector function $h(t)$ in $\mathbb{R}^n$, the following inequality is satisfied

$$i_0D_0^\alpha \left[ |h^T(t)Ph(t)| \right] \leq 2h^T(t)P\frac{\partial h}{\partial t}(t), \quad t \geq t_0,$$

where $0 < \alpha \leq 1$ and $P \in \mathbb{R}^{n \times n}$ is positive definite matrix.

**Lemma 4** ([41]). For any vectors $A, B \in \mathbb{R}^n$, there exists

$$2A^TB \leq \beta A^TA + \frac{1}{\beta}B^TB,$$

for any $\beta > 0$.

### 3. Problem statement

Consider a group of $N$ agents labeled as $1, 2, \ldots, N$. For notation convenience, $i_0D_0^\alpha f(t)$ is denoted by $D_0^\alpha f(t)$ when $i_0 = 0$. The dynamics of each agent can be expressed as

$$D_0^\alpha x_i = Ax_i + Br(t) + E_{i0}, \quad i = 1, 2, \ldots, N,$$

where $\alpha \in (0, 1)$, $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are the state vector and the input vector of agent $i$, respectively, and $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ is a saturation function satisfying $\sigma(u_i) = \text{sat}(u_{i1}) \times \ldots \times \text{sat}(u_{im})^T$, $\text{sat}(u_{ij}) = \text{sgn}(u_{ij}) \min\{|u_{ij}|, \Delta\}$ for some constant $\Delta > 0$, and $\omega_i \in \mathbb{R}^m$ is the external disturbance. The dynamics of the leader can be described as:

$$D_0^\alpha x_0 = Ax_0.$$

**Assumption 1.** For a constant $\alpha > 0$, the external disturbance $\omega_0(t)$ satisfies $\|\omega_0(t)\| \leq I$, $i = 1, 2, \ldots, N$.

**Assumption 2.** The pair $(A, B)$ is asymptotically null controllable with bounded controls, that is,

1. $(A, B)$ is stabilizable;
2. All the eigenvalues of $A$ are in the closed left-half s-plane.

**Lemma 5** ([42]). Suppose that Assumption 3 holds. Then, for each $\varepsilon \in (0, 1]$, there exists a unique matrix $P(\varepsilon) > 0$ to solve the ARE

$$A^T P(\varepsilon) + P(\varepsilon) A - 2\gamma P(\varepsilon)BB^T P(\varepsilon) + \varepsilon I = 0,$$

where $\gamma$ is a positive constant satisfying $\gamma \leq \lambda_{\max}(L + H)$ and $L$ is used to evaluate the following fractional-order derivative of the Lyapunov function. From Lemma 5, a unique positive solution $R(\varepsilon)$ for (3) exists.

**Step 2.** Present a linear feedback control law for agent $i$:

$$u_i = -B^T P(\varepsilon) \left( \sum_{j=1}^{N} a_{ij} (x_i - x_j) + h_i (x_i - x_0) \right), \quad i = 1, 2, \ldots, N.$$  

where the control can be made arbitrarily small by decreasing the value of the low gain parameter $\varepsilon$.

**Theorem 1.** Suppose that Assumptions 1–3 hold. Then, the fractional-order multi-agent systems (1) and (2) with the control input (4) can achieve semi-global robust consensus. That is to say, for any prior given bounded set $\chi \subset \mathbb{R}^n$, there exists an $\varepsilon_0 > 0$ such that, for each given $\varepsilon \in (0, \varepsilon_0]$, we have

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| \leq \frac{\sqrt{N \lambda_{\max}}}{\lambda_{\min} \cdot (\varepsilon - \beta) \cdot \beta} \|I_N \otimes (P(\varepsilon) E)\| \cdot I, \quad i = 1, 2, \ldots, N,$$

as long as all $x_i(0) \in \chi$, $i = 1, 2, \ldots, N$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of matrix $(I_N \otimes P(\varepsilon))$ and $\beta$ is a scalar satisfying $0 < \beta < \varepsilon$.

**Proof.** Define the state difference between the agent $i$ and the leader as $\tilde{x}_i = x_i - x_0$. Then, the control algorithms (4) can be rewritten as

$$u_i = -B^T P(\varepsilon) \left( \sum_{j=1}^{N} a_{ij} (\tilde{x}_i - \tilde{x}_j) + h_i \tilde{x}_i \right), \quad i = 1, 2, \ldots, N,$$

which can be written as the following compact form:

$$u = -(L + H) \otimes (B^T P(\varepsilon)) \tilde{x}.$$  

For notation convenience, define $\tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \ldots, \tilde{x}_N^T]^T$, $\tilde{\omega} = [\tilde{\omega}_1^T, \tilde{\omega}_2^T, \ldots, \tilde{\omega}_N^T]^T$. According to the Eqs. (1), (2) and (4), the following equality can be obtained:

$$D_0^\alpha \tilde{x} = (I_N \otimes A) \tilde{x} + (I_N \otimes B) \times \sigma(-(L + H) \otimes (B^T P(\varepsilon)) \tilde{x}) + (I_N \otimes E) \tilde{\omega}.$$  

Inspired by [42], firstly, the fractional-order multi-agent system (6) without saturation is considered, i.e.,

$$D_0^\alpha \tilde{x} = [I_N \otimes A - (L + H) \otimes (B^T P(\varepsilon))] \tilde{x} + (I_N \otimes E) \tilde{\omega}.$$  

Let us choose the Lyapunov function

$$V(\tilde{x}) = \tilde{x}^T (I_N \otimes P(\varepsilon)) \tilde{x}.$$  

Consequently, the fractional-order derivative of $V(\tilde{x})$ can be evaluated as

$$D_0^\alpha V(\tilde{x}) = D_0^\alpha \tilde{x}^T (I_N \otimes P(\varepsilon)) \tilde{x}.$$  

According to Lemma 3, it can be obtained that

$$D_0^\alpha V(\tilde{x}) = \frac{D_0^\alpha \tilde{x}^T (I_N \otimes P(\varepsilon)) \tilde{x}}{D_0^\alpha \tilde{x}^T (I_N \otimes P(\varepsilon)) \tilde{x}} \tilde{x} + 2 \tilde{x}^T (I_N \otimes P(\varepsilon)) (I_N \otimes A - (L + H) \otimes (B^T P(\varepsilon))) \tilde{x} + (I_N \otimes E) \tilde{\omega}.$$
\(\mathbf{BB}^{T}(\epsilon)\mathbf{x} + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} = 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} - 2(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}) + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{\delta})\mathbf{E}
\)

It should be noted that \(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}\) is a scalar function, then
\[
2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} + \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}^\mathbf{T} = \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} + \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}
\]

Then, substituting Eq. (10) into Eq. (9), it can be obtained that
\[
D^\gamma V(\mathbf{\tilde{x}}) \leq \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{A} + \mathbf{A}^T P(\epsilon)) \mathbf{\delta} - 2(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}) + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{\delta})\mathbf{E}
\]

In view of Eq. (3), thus
\[
\mathbf{\tilde{x}}[\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{A} + \mathbf{A}^T P(\epsilon)] - 2(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}) + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{\delta})\mathbf{E}
\]

Now, considering Lemma 4, the following inequality is satisfied:
\[
D^\gamma V(\mathbf{\tilde{x}}) \leq -\varepsilon \|\mathbf{\tilde{x}}\|^2 + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} - 2(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}) + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{\delta})\mathbf{E}
\]

The next thing to do is to prove the robust consensus of the fractional-order multi-agent system. In view of Eq. (8), denoting \(\lambda_{\max}(\mathbf{I}_N \otimes (P(\epsilon)))\) as \(\lambda_{\max}\), it can be obtained that \(\|\mathbf{\tilde{x}}\|^2 \leq \frac{1}{\lambda_{\max}} V(x)\).

Taking Eq. (14) into consideration, if \(\varepsilon > \beta\) exists, it can be derived that
\[
D^\gamma V(\mathbf{\tilde{x}}) \leq -\varepsilon \|\mathbf{\tilde{x}}\|^2 + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} - 2(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}) + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{\delta})\mathbf{E}
\]

Due to the fact that \(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}\) is a scalar function, then
\[
2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} + \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}^\mathbf{T} = \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} + \mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}
\]

Then, substituting Eqs. (12) and (13) into Eq. (11), it can be obtained that
\[
D^\gamma V(\mathbf{\tilde{x}}) \leq -\varepsilon \|\mathbf{\tilde{x}}\|^2 + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta} - 2(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A))\mathbf{\delta}) + 2\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon)A)\mathbf{\delta})\mathbf{E}
\]

With the inverse Laplace transform, it can be obtained that
\[
f(t) = f(0)E_\alpha \left( -\frac{t^\alpha}{\lambda_{\max}} \right)
\]

Because \(t^\alpha - 1\) and \(E_\alpha \left( -\frac{t^\alpha}{\lambda_{\max}} \right)\) are nonnegative functions, consequently the following inequality is satisfied:
\[
V(t) \leq \frac{\lambda_{\max}}{e - \beta} \|\mathbf{I}_N \otimes (P(\epsilon))\|^2 \cdot \| J_\alpha \otimes (P(\epsilon)) \|^2
\]

Moreover, letting \(\lambda_{\min} = \lambda_{\min} \mathbf{I}_N \otimes (P(\epsilon))\), it is simple to prove that
\[
V(t) \leq \frac{\lambda_{\min}}{e - \beta} \cdot \mathbf{I}_N \otimes (P(\epsilon)) \cdot \mathbf{I}_N \otimes (P(\epsilon)) \cdot \mathbf{I}_N \otimes (P(\epsilon))
\]

In other words, it implies that
\[
\lim_{t \to \infty} \|\mathbf{\tilde{x}}\|^2 = \|\mathbf{\tilde{x}}\|^2 \leq \sqrt{\frac{N\lambda_{\max}}{\lambda_{\min}} \|\mathbf{\tilde{x}}\|^2 \cdot \mathbf{I}_N \otimes (P(\epsilon)) \cdot \mathbf{I}_N \otimes (P(\epsilon)) \cdot \mathbf{I}_N \otimes (P(\epsilon))}
\]

which shows that the fractional-order multi-agent system achieve the robust consensus when the input saturation is not taken into consideration.

It remains to show that when input saturation exists, the similar results can be deduced. Accordingly, low-gain feedback is utilized to consider the case when input saturation exists. Lemma 2.3.6 in [42] reveals that for any given bounded set of initial conditions, the control can be made arbitrarily small by reducing \(\varepsilon\). Therefore, it can be guaranteed that
\[
\left\| \beta^T P(\epsilon) \left( \sum_{j=1}^{N} a_{ij}(\tilde{x}_i - \tilde{x}_j) + h_i \tilde{x}_j \right) \right\|_\infty \leq \Delta, \quad i = 1, 2, \ldots, N,
\]

hold by adjusting the low gain parameter \(\varepsilon\). Then, saturation nonlinearity will remain linear in this way. From what have been stated above, there always exists a low gain parameter \(\varepsilon\) to transform the fractional-order multi-agent system (6) with input saturation into (7) without input saturation.

The next thing to do is to find a set guaranteeing that in this set, the Lyapunov function \(\mathbf{\tilde{x}}(\mathbf{I}_N \otimes (P(\epsilon))\mathbf{\tilde{x}}\) presented in the case when saturation is not considered always exists. Then, according to the above proof, the robust consensus can be achieved in this set. In other words, semi-global robust consensus can be achieved. According to Eq. (16), it can be obtained that
\[
V(t) \leq \frac{\lambda_{\max}}{e - \beta} \|\mathbf{I}_N \otimes (P(\epsilon))\|^2 + f(0)E_\alpha \left( -\frac{t^\alpha}{\lambda_{\max}} \right)
\]

Moreover, according to Lemma 2, it’s easy to see that
\[
E_\alpha \left( \frac{t^\alpha}{\lambda_{\max}} \right) \leq d \quad \text{where} \quad d \quad \text{is a positive constant and when (19)}
\]
\text{Let } V(t) \text{ and } V(0) \text{ can be rewritten as } V(\bar{x}) \text{ and } V(\bar{x}(0)), \text{ respectively. It can be seen } V(\bar{x}) \text{ has an upper bound determined by } V(\bar{x}(0)). \text{ From the related definition, it can be derived } V(\bar{x}(0)) = \bar{x}(0)^T I_0 (\|I_0 \otimes (P(\epsilon)E)\| )^2.

where \( V(t) \) and \( V(0) \) can be rewritten as \( V(\bar{x}) \) and \( V(\bar{x}(0)) \), respectively. It can be seen \( V(\bar{x}) \) has an upper bound determined by \( V(\bar{x}(0)) \). From the related definition, it can be derived \( V(\bar{x}(0)) = \bar{x}(0)^T I_0 (\|I_0 \otimes (P(\epsilon)E)\| )^2 \).

Owing to the fact that for any prior given bounded \( x_i(0) \), \( x_i(0) \in \chi \), \( i = 0, 1, 2, \ldots, N \) is bounded and \( \lim \epsilon = 0 \). Then, for any prior given bounded set \( \chi \), \( V(\bar{x}(0)) \) is bounded and there always exist a constant \( c \)

\[
\epsilon \geq \epsilon V(0) + (1 - \epsilon) \frac{\frac{\lambda_{\text{max}}}{\epsilon - \beta}}{\sum_{\epsilon - \beta}} N \|I_0 \otimes (P(\epsilon)E)\|^2. \tag{19}
\]

Let \( L_\epsilon(c) = \{ \bar{x} \in \mathbb{R}^N : V(\bar{x}) \leq c \} \). Therefore, the Lyapunov function always exists in the set \( L_\epsilon(c) \). Then, for \( \bar{x} \in L_\epsilon(c) \), an \( \epsilon^* \) can be obtained by solving

\[
\left\| B^T P(\epsilon) \left( \sum_{i=1}^{N} D_{ij}(\hat{x}_i - \bar{x}_j) + h_i \bar{x}_i \right) \right\| \leq \Delta, \quad i = 1, 2, \ldots, N. \tag{20}
\]

Finally, for any prior given bounded sets \( \chi \subset \mathbb{R}^n \), there exists an \( \epsilon^* > 0 \) such that, for each given \( \epsilon \in [0, \epsilon^*] \),

\[
\lim_{t \to \infty} \|\bar{x}_i(t) - x_i(0)\| \leq \sqrt{\frac{\lambda_{\text{min}}}{\lambda_{\text{max}}} \cdot (\epsilon - \beta)} \cdot \|I_0 \otimes (P(\epsilon)E)\| \cdot L, \quad i = 1, 2, \ldots, N,
\]

as long as \( x_i(0) \in \chi \) for all \( i = 0, 1, 2, \ldots, N \). That is, the fractional-order multi-agent systems (1) and (2) with the control law (4) can achieve semi-global robust consensus. This completes the proof.

**Remark 1.** When input saturation is considered in studying the consensus of fractional-order multi-agent systems, an important problem is to find the invariant set \( L_\epsilon(c) \). In this set, saturation nonlinearity can be eliminated by choosing a proper low-gain parameter. Moreover, the consensus can be achieved in this set after the input saturation is eliminated. It should be noted that the consensus is semi-global because the set of initial states is bounded.

**Remark 2.** In [43], low gain feedback is also utilized to achieve the consensus of multi-agent systems with input saturation. Compared with [43], because the Leibnizt rule is not applicable for the fractional-order derivative, it makes trouble to evaluate the evolution of Lyapunov function \( V(t) \) with time \( t \) and then it’s more difficult to find the invariant set \( L_\epsilon(c) \) and achieve the consensus within this set. In addition, leader-following consensus can be achieved in [43] while in this paper, in consideration of external disturbances, robust consensus is obtained consequently.

**Theorem 1** shows that the existence of the linear feedback control law (4) for a prior given bounded set \( \chi \) of the initial states. However, how to determine the control gain \( P(\epsilon) \) remains unclear. In the following, we will show the way to find the feasible solution of \( \epsilon^* \) which will directly lead to the solution \( P(\epsilon) \) of (3) with \( \epsilon \in (0, \epsilon^*) \). It’s obvious that if we find \( k = \max\{\|\bar{x}_i\|\} \) for \( \bar{x}_i \in L_\epsilon(c) \), i.e., \( \bar{x}_i \neq P(\epsilon)^* \bar{x}_i \leq c \), then

\[
\left\| B^T P(\epsilon) \left( \sum_{i=1}^{N} D_{ij}(\hat{x}_i - \bar{x}_j) + h_i \bar{x}_i \right) \right\| \leq \left\| (2N + 1)kB^T P(\epsilon^*)1 \right\| \infty
\]

holds, as a result, if \( \left\| (2N + 1)kB^T P(\epsilon^*)1 \right\| \infty \leq \Delta \) holds, the feedback control remains linear and the conclusion in **Theorem 1** holds.
and let the fractional order $\alpha = 0.9$. It is easy to verify that $(A, B)$ is asymptotically null controllable with bounded controls. The interaction graph among the agents is given by Fig. 1.

It’s obvious that the topology of the graph satisfies Assumption 3. In addition, the following matrices are obtained from Fig. 1:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$H = \text{diag}(1.1, 0, 0).$$

Then, $\lambda_{\text{min}}(L + H) = 1$, so choose $\gamma = 0.9$. Choose a set $[-10, 10] \times [-10, 10]$ as the set $\mathcal{X}$ of initial states and then $a = 10$. The external disturbance are given as: $\omega_1 = 1.1 \sin(3t - 1)$, $\omega_2 = -\cos(2t)$, $\omega_3 = 0.8 \sin(t) + 0.7 \cos t$, $\omega_4 = 0.2 \cos(11t - 4)$. From Assumption 2, $l = 1.1$. Let $\Delta = 25$. The parameter $\epsilon^*$ can be calculated by using Algorithm 1.

Step 1: Choose $\epsilon^* = 0.4$.

Step 2: Firstly, from (3), calculate

$$P(0.4) = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

then it can be obtained that $\lambda_{\text{max}} = 0.4714$. Secondly, the positive constant $d$ in Lemma 2 is chosen as $d = 0.001$. Substituting $a$, $\epsilon^*$, $P(\epsilon^*)$, $\lambda_{\text{max}}$ and $d$ into (21), it can be obtained that $c = 16.041$ and then $k = 4.888$.

Step 3: It can be calculated that

$$\left\| (2N + 1)kB^TP(\epsilon^*)I \right\|_{\infty} = 20.74 \leq \Delta.$$

so the parameter $\epsilon^* = 0.4$ is feasible. Therefore, any $\epsilon \in (0, 0.4]$ can be chosen to construct the linear feedback control law so that the consensus can be achieved.

- The following case is to verify Theorem 1.

When $\epsilon = 0.1$, according to Eq. (3), it can be obtained that

$$P(0.1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2375 \end{bmatrix}.$$ 

As the set of initial states is $[-10, 10] \times [-10, 10]$, the initial values of the leader and the four followers can be chosen as:

$$x_0(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad x_1(0) = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \quad x_4(0) = \begin{bmatrix} 10 \\ 6 \end{bmatrix}.$$ 

It’s obvious that all the initial states are within the set $[-10, 10] \times [-10, 10]$. The simulation result is shown in Fig. 2 to verify Theorem 1 in Section 4. Fig. 2 shows that when there exist input saturation and external disturbances simultaneously, state errors asymptotically converge to a range. That is, the fractional-order multi-agent systems (1) and (2) can achieve robust consensus. Therefore, the effectiveness of Theorem 1 is verified.

- The following case is to verify Corollary 1 and to show the effect of $\epsilon$ on the convergence rate:

Consider the case when there only exists input saturation while external disturbance does not exist. $\epsilon = 0.1$ and 0.01 are chosen in this case. Firstly, the error $e_i = \|x_i\|_2$ is defined to evaluate the state errors of agent $i$ and when $\epsilon = 0.01$.

$$P(0.01) = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.0745 \end{bmatrix}.$$ 

The aforementioned initial states are also chosen in this case and the simulation result is shown in Fig. 3. From Fig. 3, it can be seen that when there only exists input saturation, state errors asymptotically converge to zero. That is, the fractional-order multi-agent systems (22) and (23) can achieve leader-following consensus. Therefore, the effectiveness of Corollary 1 is verified.

Moreover, $\epsilon$ is 0.1 in Fig. 3(a) and 0.01 in Fig. 3(b). Note that the convergence speed of the agent in Fig. 3(a) is faster than that in
The error $e$ with input saturation for different $\varepsilon$.

The initial state is within $\mathcal{X}$ and the initial state is beyond $\mathcal{X}$. The error $e$ with input saturation for different initial state.

6. Conclusions

In this paper, robust consensus of fractional-order multi-agent systems with input saturation and external disturbances is investigated. By introducing a priori bounded set, the follower agent remains linear within this set. Mittag-Leffler stability theory and low gain feedback technique are utilized to solve the consensus problem. It should be noted that robust consensus can be obtained when input saturation and external disturbances exists simultaneously. Moreover, as an exceptional case, leader-following consensus can be obtained when there only exists input saturation. An interesting future topic is to investigate the fractional-order nonlinear multi-agent systems in the presence of the input saturation and external disturbances.
Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grants 61773172, 61572210, and 51537003, the Natural Science Foundation of Hubei Province of China (2017CFA035) and the academic frontier youth team of HUST.

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